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MONOPOLE INSTABILITY I : NEGATIVE MODES

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Abstract :

For monopoles with non-vanishing Higgs potential it is shown that with respect to (asymptotic) variations of the gauge potential (a) the stability problem reduces to that of a pure gauge theory on the 2-sphere (b) each topological section admits one, and only one, stable monopole, and (c) each non-stable monopole admits $2\,\Sigma\,(2(q)-1)$ negative modes where the sum goes over all negative eigenvalues q of the non-Abelian charge Q. An explicit construction for (i) the unique stable monopole (ii) the negative modes and (iii) the spectrum of the Hessian, on the 2-sphere, is then given. The relationship of the instability-index $(2(q)-1)$ to the Atiyah-Singer and Witten indices is exhibited and the general theory is illustrated for the little groups U(2), U(3) and O(5).

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1. Introduction

By linearizing the field equations around a monopole solution, Brandt and Neri [1] and Coleman [2] have shown that most non-Abelian monopoles are unstable with respect to small perturbations. Using the properties of the rotation group, they demonstrate in fact that a monopole can be stable only if all eigenvalues of its non-Abelian charge [3] Q satisfy the "Brandt-Neri" condition

$$q = 0 \text{ or } \pm 1/2. \tag{1.1}$$

Goddard and Olive [4] prove then that the semisimple part of Q must be of a very special form, known in representation theory as a "minimal co-weight" (see Sections 2 and 4 for details).

As it is well-known, monopoles fall into topological sectors separated by infinite - energy barriers, and one can prove [2,4,5] that each such topological sector there is a unique stable monopole.

Asymptotic monopole configurations with residual group H behave very much like pure Yang - Mills theory on S^2 with gauge group H. But this is a special case of YM on Riemann surfaces, studied by Atiyah and Bott [6]. Translating the topological formula of [6], Friedrich and Habermann [7] have shown that such a configuration (also characterized by Q) admits

$$\nu = 2 \sum 2|q|-1 \tag{1.2}$$

negative modes, where the sum goes over all negative eigenvalues of Q. (1.2) obviously implies the BN condition (1.1).

The aim of this paper is to relate and complete the above results. After summarizing the necessary algebraic tools, we review those properties of finite-energy configurations (Section 3) and of solutions (Section 4) which are relevant for our purposes. Most of the contents of these sections are already known [2,8], (a possible exception being the explicit formula (4.10) for the unique stable monopole of a given topological sector) but we have assembled the results from different sources and summarized them for completeness and for the convenience of the reader.

The monopole is stable if its second variation (called the <u>Hessian</u>) has no negative eigenvalues. In Section 5 we show first that, for non-zero Higgs potentials and suitable variations of the gauge field, the 3-dimensional problem essentially reduces to pure YM theory on $\mathbf{S}^2$. (Although the radial contribution is not zero, it yields a "mass" term which does not change the stability behaviour.

For $\mathbf{S}^2$ the general theory of Atiyah and Bott [6] can be related to the Brandt-Neri-Coleman rotation - group approach. Indeed, on the q-eigenspace the interesting part of the Hessian is

$$\int dr d\Omega \, \mathrm{tr}\, \{ (\mathbf{J}^2 - q(q-1))\delta\mathbf{A}, \delta\mathbf{A}\} \; + \; q \int dr d\Omega \, \mathrm{tr}(\delta\mathbf{A})^2, \tag{1.3}$$

where r and $\Omega = (\theta,\varphi)$ are polar coordinates and $\mathbf{J}^2 = j(j+1)$ is the <u>Casimir</u> of the angular momentum vector $\mathbf{J}$. Since the first term is non-negative and the first non-zero eigenvalue is at least $2|q|$, negative mode can occur only if the first term in (1.3) vanishes, which happens only if

$$|q| \leqslant -1 \quad \text{and} \quad j = |q| - 1. \tag{1.4}$$

From this result it is evident that the negative modes form a $2j+1$ = <u>$2|q|-1$ dimensional SU(2) - multiplet</u>. A simple way of counting the number of negative modes is to use the diagram introduced by Bott [9].

The special form (1.3) of the Hessian makes it possible to construct the negative modes explicitly and in terms of the stereographic (complex) coordinate z on $\mathbf{S}^2$ and in section 6 they are shown to be

$$\delta \mathbf{A} = \bar{z}^{k}/(1+z\bar{z})^{|q|}, \qquad k = 0,\dots,2|q|-2 \tag{1.5}$$

Summing over all eigenvalues one gets once more the index formula (1.2). The positive modes, or $j \geqslant |q|$ states, may be constructed by the same technique.

Not surprisingly, the expressions in (1.5) are known to generate the spin $2|q|-2$ representations of the rotation group. (The (−2) comes from the fact that the integrand in the first term in (1.3) is a combination of a spin $2|q|$ and of a spin 1 field [2]).

For Bogomolny-Prasad Sommerfield monopoles [10,11] there is an extra term q^2 in the Hessian due to the long-range Higgs field (see (3.6)), which cancels the corresponding term in (1.3) and we get rather

$$\delta^2 E = \int dr\, d\Omega\, \mathrm{tr}\{ (m^2 + \mathbf{J}^2)\, \delta\mathbf{A}, \delta\mathbf{A}\}, \tag{1.6}$$

which is manifestly positive. It follows that BPS monopoles are stable with respect to variations of the gauge field. The instabilities found by Taubes [12] arise because he includes variations $\delta\Phi \neq 0$ of the Higgs field.

In Section 7 we illustrate our general theory . First we study the situation with residual symmetry groups $H = U(2)$ and $H = U(3)$. However the simplest example where the special property of the stable charges (mentioned above) enters, is when the semisimple part of H is (a covering of) SO(5). So we have studied this case also.

The monopole-stability problem is related to that of loop-stability in the residual group [6,7]. This and the topological aspects of monopole instabilities will be discussed in a forthcoming paper.

2. Algebraic Structure [13]

Let us consider a compact simple Lie algebra $\mathfrak{h}$ and choose a Cartan subalgebra $\mathfrak{t}$. A root α is a linear function on the complexified Cartan algebra $\mathfrak{t}^{\mathbb{C}}$, and to each α is associated a vector E_α from $\mathfrak{h}^{\mathbb{C}}$ (the familiar step operator) which satisfies, with any vector H from $\mathfrak{t}^{\mathbb{C}}$, the relation $[H, E_\alpha] = \alpha(H)\, E_\alpha$. There exists a set of primitive roots α_i , $i = 1,\ldots r$ (= rank) such that every positive root is a linear combination of the α_i with non-negative integer coefficients i.e. $\alpha = \sum m_i\, \alpha_i$ for all α.

Alternatively, we can consider the real combinations $X_\alpha = E_\alpha + E_{-\alpha}$ and $Y_\alpha = i\,(E_\alpha - E_{-\alpha})$ which satisfy

$$[H, X_\alpha] = q_\alpha Y_\alpha \qquad \text{and} \quad [H, Y_\alpha] = -q_\alpha X_\alpha \tag{2.1}$$

where $q_\alpha = \alpha(Q)/i$ is real.

If α is a root, define the vector H_α in $\mathfrak{t}^{\mathbb{C}}$ by $\alpha(X) = \mathrm{tr}(H_\alpha X)$. Chosing the normalization $\mathrm{tr}(E_\alpha, E_{-\alpha}) = 1$, we have $[E_\alpha, E_{-\alpha}] = H_\alpha$. Let us now define the <u>primitive</u> <u>charge</u> Q_i by

$$Q_i = 2H_i/\mathrm{tr}(H_i^2). \tag{2.2}$$

The primitive charges form a natural (non-orthogonal) basis for the Cartan algebra and by adding the E_α's we get a basis for the Lie algebra $\mathfrak{h}^{\mathbb{C}}$. Similarly, the primitive charges and the $\{X_\alpha, Y_\alpha\}$ form a basis for the real algebra $\mathfrak{h}$. For H = SU(2) for example, the Cartan algebra can be taken to consist of diagonal matrices (multiples of σ_3) , the $E_\pm$ are the familiar step operators, and $X = \sigma_1$ and $Y = \sigma_2$.

Denote by H the (unique) compact, simple and simply connected Lie group generated by $\mathfrak{h}$. All the primitive charges satisfy the quantization condition $\exp 2\pi i Q_i = 1$, where the exponential is taken in the fundamental

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Denote by H the (unique) compact, simple and simply connected Lie group generated by $\mathfrak{h}$. All the primitive charges satisfy the quantization condition $\exp 2\pi i Q_i = 1$, where the exponential is taken in the fundamental representations i.e. in H. (in the case of the orthogonal groups the word fundamental denotes the n - dimensional (vector) representation for SO(n) itself and the fundamental spinor representation(s) for Spin (n)).

If a general charge Q is defined to be any element of the Cartan algebra that is similarly quantized in the fundamental representation i.e.

$$exp\ 2\pi i Q = 1, \tag{2.3}$$

then it can be expanded as a linear combination ot the primitive Q_i 's with integer coefficients, $Q = \sum n_i Q_i$, n_i an integer. Since the primitive charges may be regarded as (non-orthogonal) base vectors in r - dimensional euclidean space, the set of all charges may be regarded as a lattice generated by the Q_i. This lattice is usually called the charge lattice and is denoted by Γ_Q. As it will be explained later, monopoles are charges and the instabilities lie in suitable E_α (or X_α, Y_α) directions.

Let us introduce a dual basis for the Cartan algebra with elements W_i where

$$\alpha_i(W_j) = Tr(W_i H_j) = \delta_{ij}\,, \quad i\,,\ j\ = 1\,, \ldots r\ . \tag{2.4}$$

By comparing (2.4) with the conventional definition for primitive weights [13], for which there is an extra factor $(\alpha_i, \alpha_i)/2$ in front of the δ_{ij}, one sees that the W_i 's are just re-scaled weights. They are called co-weights [4] and it is evident that they can be normalized so as to coincide with weights (by chosing $(\alpha_i, \alpha_i) = 2$) for all groups whose roots are all of the same length i.e. all groups except Sympl (2n), Spin (2n+1) and G_2. The integer combinations $\sum m_i W_i$ form a lattice that we denote by Γ_W and since all roots take integer values on a charge, the W - lattice actually contains the charge lattice, $\Gamma_W \supset \Gamma_Q$.

The root - planes of $\hbar$ are those vectors X in the Cartan algebra for which α(X) is an integer. The W - lattice containing the charge lattice, together with the root planes, forms the (Bott [9]) diagram of H.

Finally, although, in general, $exp\ 2\pi i W_i$ is not unity in the fundamental representation, it is unity in the adjoint representation and hence

$$exp\ 2\pi i W_j = z_j \tag{2.5}$$

belongs to the centre Z of H. Note, however, that the correspondence W $\sim$ z is one-to-one only for SU(n), since for the other groups there are r W 's but less then r elements in Z (as shown in Table 1.).

On the other hand the correspondence W $\sim$ z can be made one-to-one by restricting the W's to those ones, $\mathring{W}$'s say, for which the geodesics $exp\, 2\pi it\,\mathring{W}$ for $0 \leq t \leq 1$ are geodesics of minimal length from $\mathbf{1}$ to z i.e. for which $\mathrm{Tr}\, W^2$ is minimal for each z ϵ Z. (Since the weights W are all of different length and are unique up to conjugation, the $\mathring{W}$ for each z ϵ Z will be unique up to conjugation). Such co-weights are called minimal co-weights [4] and a simple intuitive way to find them (indeed an alternative way to introduce them) is as follows:

Let z ϵ Z be a central element in the fundamental representation F of the group and let f be the dimension of F. Then by Schur 's lemma and the unimodularity of F the elements z must be of the form $z = (exp\, 2\pi i\lambda)1_f$, where λ = p/f and p is an integer between 0 and f, $0 \leq p \leq f$. (Note that if F is real or pseudo - real z must be real and therefore equal ± 1, a result which expains the abundance of $Z = 1_f$ and $Z = \mathbf{Z}_2$ in Table 1). It is clear that z is an element in the centre of SU(f) as well as G, and hence one may start by constructing the minimal geodesic from $\mathbf{1}_f$ to z in SU(f). Let this be $exp\, 2\pi it\Sigma$, where Σ is a generator of SU(f) and $0 \leq t \leq 1$. Since $exp\, 2\pi i\Sigma = (exp\, 2\pi i\lambda)1_f$, the eigenvalues of Σ can only be of the form $\lambda + l_k$, k = 1,..,f where the l_k are integers, and hence the geodesic length must be proportional to $\sum_k(\lambda + l_k)^2$. It is clear that this length will be smaller for $l_i = 0 \; or \; -1$ than for any other set of l's. But since Σ must be traceless, there is actually only one Σ for which $l_k = 0, -1$, namely

$$\Sigma = \frac{1}{f}\begin{pmatrix} p1_q & \\ & -q1_p \end{pmatrix} \tag{2.6}$$

(up to conjugation). For G = SU(n) this is evidently the end of the story since n = f and hence $\mathring{W} = \Sigma$. But the remarkable fact is that for the other groups also, it is the end of the story. More precisely, for every group G in Table 1 , $\mathring{W}$ is an SU(f) - conjugate of Σ. We do not know of any universal (G - independent) proof of this result, but it is not difficult to verify it for each class of group in Table 1 separately. For this purpose it is convenient to characterize Σ in a conjugation - independent manner, namely to write

$$(\Sigma - \frac{p}{f}1)(\Sigma + \frac{q}{f}1) = 0\,, \tag{2.7}$$

since then one has only to verify that the group in question has a generator satisfying (2.7) for a given central element i.e. given fraction p/f. Now for the groups with centre $\mathbf{Z}_2$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$ equation (2.7) reduces to $\Sigma^2 = 1/4$ and it is easy to verify that the generators shown in Table 1 have this property. Similarly for the only group with centre $\mathbf{Z}_3$, namely E_6, it can be verified directly that it has a generator of the form $(y/3) \times 1_q$ and that such a generator satisfies (2.7) for p/f = 1/3. The class of group with centre $\mathbf{Z}_4$, namely Spin (4n+2), is perhaps the most interesting. In this case Σ should satisfy the equations

$$\Sigma^2 = \frac{1}{4} \quad or \quad \left(\Sigma \pm \frac{1}{4}1\right)\left(\Sigma \mp \frac{3}{4}1\right) = 0 \tag{2.8}$$

and one can see that the entries for $\mathring{W}$ given in Table 1 satisfy these equations and are generators by recalling that Spin(4n+2) splits into the direct sum of the two inequivalent spin representations of Spin(4n) with generators $\frac{1}{2}(1 \pm \gamma)[\gamma_\mu, \gamma_\nu]$ repsectively, where $\gamma = \gamma_1\gamma_2...\gamma_{4n}$ is the generalization of γ_5 to 4n dimensions.

Collecting the results for the different groups G together, one sees that in all cases the $\mathring{W}$ in the fundamental representation are matrices with (i) only two distinct eigenvalues and (ii) unit difference between eigenvalues. Since it can be shown that the converse is also true (any such matrix is an $\mathring{W}$) the $\mathring{W}$ may actually be characterized by this property. Furthermore, since the adjoint representation occurs in the tensor product $F \times F^*$ the property (i), (ii) may also be expressed by saying that the $\mathring{W}$'s can have only eigenvalues 0 or $\pm$ 1 in the adjoint representation, and since the converse is again true, the $\mathring{W}$' s may be characterized by this (0, $\pm$ 1) property also.

In terms of the roots α, the (0, $\pm$ 1) property may be expressed by saying that for any positive root α the quantity $\alpha(\mathring{W})$ must be zero or unity i.e. $[E_\alpha, \mathring{W}] = \alpha(\mathring{W})E_\alpha \Rightarrow$

$$\alpha(\mathring{W}) = 0, \pm 1 \quad for\ all\ \alpha > 0 \tag{2.9}$$

cf. [4,5]. Viewed on the (Bott) diagram, those vectors satisfying this condition either lie themselves in the centre or belong to the root plane which is the closest to the centre. Examples are given in Section 7.

If one considers in particular the expansion of the highest root θ in terms of the r primitive roots α_i, $\theta = \sum h_i\alpha_i$, $h_i \geq 1$, and applies (2.9) to both sides of this equation, one sees that $\alpha_i(\mathring{W})$ can be non-zero for only one primitive root, $\mathring{\alpha}_i$ say, and that the coefficient $\mathring{h}_i$ of $\mathring{\alpha}_i$

must be unity. This result provides us with a simple, practical method of identifying the W in terms of primitive weights, namely as the duals of those primitive roots for which the coefficient in the expansion of θ is unity [4,5] and this method has been used to obtain the identifications given in Table 1.

3. Finite energy configurations and Higgs breakdown [8]

Our starting point is a static, purely-magnetic Yang-Mills Higgs (YMH) system with a simple and compact gauge group G, given by the Hamiltonian

$$2E= \int d^3x \{ \mathrm{tr}\, \mathbf{B}^2 + (\mathbf{D}\Phi,\mathbf{D}\Phi) + 2V(\Phi)\}, \quad V(\Phi) \geqslant 0; \tag{3.1}$$

where $V(\Phi)$ is a Higgs potential for the scalar field Φ, $\mathbf{B}$ is the Yang-Mills magnetic field and $\mathbf{D}\Phi$ is the covariant derivative, $B_i = \varepsilon_{ijk}B_{jk}/2$, $B_{jk} = \nabla_j A_k - \nabla_k A_j + [A_j,A_k]$, $D_i\Phi = \nabla_i\Phi + A_i\Phi$ where $\mathbf{A}$ is the gauge potential and $\mathbf{A}\Phi$ denotes its action on Φ in the representation to which Φ belongs.

In this section we shall not require that the fields satisfy the Euler – Lagrange field equations, but only that they be of finite energy i.e. such that the integral in (3.1) converges. One reason for this is to emphasise that the most important spontaneous symmetry breakdown, namely that of the Higgs potential, comes from the finite-energy and not from the field equations.

We shall consider the three terms in the Hamiltonian (3.1) in turn. It will be convenient to use the radial gauge $\mathbf{x}.\mathbf{A} = 0$.

Pure gauge term $\mathrm{tr}\, \mathbf{B}^2$

For sufficiently smooth gauge fields the finite-energy condition imposed by this term is evidently

$$\mathbf{A}(x) \rightarrow \mathbf{a}(\Omega)/r, \qquad \mathbf{B}(x) \rightarrow \mathbf{b}(\Omega).\mathbf{x}/r^3 = b(\Omega)/r^2, \tag{3.2}$$

where Ω denotes the polar angles (θ,φ), $b(\Omega) = (\boldsymbol{\partial}\times\mathbf{a}(\Omega) + \mathbf{a}(\Omega)\times\mathbf{a}(\Omega)) . \mathbf{x}/r$ and $\boldsymbol{\partial} = r\boldsymbol{\nabla}$.

Although $\mathbf{a}(\Omega)$ and $\mathbf{b}(\Omega)$ in must be single-valued on the sphere S^2, they need not be quantized for (3.2) to be satisfied. The situation is analogous to an Aharonov-Bohm potential in two dimensions, where the gauge-field is single-valued but the magnetic flux need not be quantized. [Only for the so-called vortex system, in which there exists, in addition to the gauge -field, a scalar field $\Phi(x)$, which remains finite and covariantly constant as $r \to \infty$ does the flux become quantized. The generalization of the vortex case will be seen below].

Higgs Potential $V(\Phi)$

The finite-energy condition for this term is evidently $r^2 V(\Phi) \to 0$ as $r \to \infty$. A necessary condition for this is that $V \to 0$ .But $V \geqslant 0$ is assumed to be a Higgs potential i.e. minimizes on a non-trivial group orbit G/H. Therefore, at large distances, the Higgs field is not zero, takes its values on the orbit G/H and may depend non-trivially on the polar angles Ω : $\Phi(r,\Omega) \to \Phi(\Omega)$ as $r \to \infty$. Then $\Phi(\Omega)$ defines a map of S^2 into the orbit G/H and thus a homotopy class in $\pi_2(G/H)$. Since this class cannot be changed by smooth deformations [8] , the space of finite-energy configurations splits into topological sectors labelled by $\pi_2(G/H)$.

The topological sectors can also be labelled by classes in $\pi_1(H)$. Indeed, on the upper and respectively lower hemispheres N and S of S^2,

$$\Phi(\Omega) = \begin{cases} g_N(\Omega)\Phi(E) & \text{in N} \\ g_S(\Omega)\Phi(E) & \text{in S} \end{cases} \tag{3.3}$$

where E is an arbitrary point in the overlap, e.g. the "east pole".

$$h(\varphi) = g_N^{-1}(\varphi) g_S(\varphi) \tag{3.4}$$

(where φ is the polar angle on the equator of S^2) is a loop in H which represents the topological sector.

For simplicity, and because it is the most relevant case, we shall assume that the homotopy group $\pi_2(G/H)$ is described by a single integer "quantum "number m. This is equivalent to assuming that the Lie algebra $\mathfrak{h}$ of H has a 1-dimensional centre generated by a vector Ψ, and H_{ss}, the semisimple subgroup of H generated by $\mathfrak{h}_{ss}=[\mathfrak{h},\mathfrak{h}]$, is simply connected.

When the Higgs field $\Phi(\Omega)$ belongs to the adjoint representation of a classical group G, and the Higgs potential $V(\Phi)$ is quartic, the one - dimensionality of the centre is actually not an assumption at all, but is simply a consequence of the fact that Michel's conjecture [14] is valid for this case. In fact, for the adjoint representation of a classical group Φ itself generates the centre and is parallel to one of the primitive (but not necessarily minimal) W_j's.

Whether $\Phi(\Omega)$ is in the adjoint or not, we can associate to it a new, adjoint " Higgs" field Ψ defined by $\Psi(\Omega) = g(\Omega)\, \Psi\, g^{-1}(\Omega)$, where $g(\Omega)$ is any of those in (3.3). $\Psi(\Omega)$ is well-defined, because Ψ belongs to the center of $\mathfrak{h}$. (Of course, if $\Phi(\Omega)$ is in the adjoint representation, then $\Psi(\Omega)$ and $\Phi(\Omega)$ are proportional).

The quantum number m can also be calculated as a surface integral. Indeed, the projection onto the centre of H of the charge-lattice Γ_Q is a 1-dimensional lattice in the centre. If we choose Ψ to be its $\mathbb{Z}$-generator then the quantum number m can be recovered [15] according to

$$2\pi m = \int d\Omega_{ij} \mathrm{tr}\{\acute{\Psi}(\Omega)\,[\partial_i\Psi(\Omega),\partial_j\Psi(\Omega)]\}\,. \tag{3.5}$$

The homotopy classification is not merely convenient, but is <u>mandatory</u> in the sense that the classes are separated by infinite energy barriers. Thus, while an interpolated field of the form $\Phi^t = t\Phi_1+(1-t)\Phi_2$, $0\leqslant t\leqslant 1$ between two finite-energy configurations Φ_1 and Φ_2 is perfectly smooth if Φ_1 and Φ_2 are smooth, it does not satisfy the finite-energy condition $r^2V(\Phi^t) \to 0$, or even $V(\Phi^t) \to 0$, as $r\to\infty$, for general t. Note that since not only $V \to 0$ but $r^3V \to 0$ one has, using the notation $\eta = \Phi(r,\Omega)$ -

$\Phi(\Omega)$, $r^3 M_{\alpha\beta}\eta_\alpha\eta_\beta$ -->0 where $M_{\alpha\beta} = \partial^2 V/\partial\Phi_\alpha\partial\Phi_\beta|_{r=0}$, and hence for generic potentials (i.e. those for which the only zeros of the 'mass – matrix' $\partial^2 V/\partial\Phi^2$ at $V = V_{min}$ are the Goldstone zeros) the physical part of η falls off faster than r^{-1} as $r \to \infty$ and one gets $\Phi(x) \to \Phi(\Omega) + \eta(r,\Omega)$, where $r\eta(r,\Omega) \to 0$ as $r \to \infty$. A notable exception to this observation is the Bogomolny – Prasad – Sommerfield (BPS) case $V = 0$, for which the Bogomolny condition $\mathbf{B} = \mathbf{D}\Phi$ implies [10,11] that

$$\Phi(x) \to \Phi(\Omega) + b(\Omega)/r + O(1/r^2) \text{ as } r \to \infty \tag{3.6}$$

<u>The cross-term $(\mathbf{D}\Phi)^2$</u>

This final term involves both Φ and $\mathbf{A}$ and it hence provides the connection between the Higgs field $\Phi(\Omega)$ and the gauge field $\mathbf{b}(\Omega)$ and thus puts a topological constraint on the gauge field. As might be expected from the vortex analogy, this constraint may be expressed as a quantization condition as follows: the finite-energy condition is easily seen to be $r^2(D\Phi)^2 \to 0 \Rightarrow$

$$\mathbf{d}\Phi \equiv \boldsymbol{\partial}\Phi + \mathbf{a}(\Omega)\Phi(\Omega) = 0 \tag{3.7}$$

and thus also $\mathbf{d}\Psi(\Omega) = \boldsymbol{\partial}\Psi(\Omega) + [\mathbf{a}(\Omega),\Psi(\Omega)] = 0$. $\Phi(\Omega)$, and $\Psi(\Omega)$ are hence both covariantly constant on $\mathbf{S}^2$. It follows from (3.7) that the topological quantum number m can also be expressed as

$$2\pi m = \int d\Omega \, tr(\Psi b) \tag{3.8}$$

Equation (3.8) is the generalizaton of the vortex quantization condition mentioned earlier and it shows that in general it is not the gauge field b itself, but only its <u>projection onto the centre</u> that is quantized. Note that the quantization of $\int tr(b\Psi)$ is again <u>mandatory</u> since the value of $tr(b,\Psi)$

cannot be changed without violating at least one of the finite-energy conditions $r^2V \to 0$ or $r^3(D\Phi)^2 \to 0$ and thus passing through an infinite energy barrier. Notice also that the value of (3.8) is actually independent of the choice of the Yang-Mills potential **A** as long as Φ is covariantly constant [15].

4. Finite-energy solutions of the field equations [3,8]

The only condition imposed on the YMH configurations $(\mathbf{A},\Phi)$ up to this point is that the energy be finite. But it is obviously of interest to consider the special case of finite-energy configurations that are also solutions of the YMH field equations,

$$D^2\Phi = \partial V/\partial\Phi \quad \text{and} \quad \mathbf{D}\times\mathbf{B} = (\Phi, \boldsymbol{\tau}\mathbf{D}\Phi)\,, \tag{4.1}$$

where $\boldsymbol{\tau}$ denotes the generators of the Lie algebra in the suitable representation.

Finite-energy solutions may be classified using data referring to the field $b(\Omega)$ alone. For this it is sufficient to consider the field equations (4.1) for large r, in which case they reduce to

$$\Delta\eta_\alpha = (\partial^2 V/\partial\Phi_\alpha\partial\Phi_\beta)\eta_\beta \quad \text{and} \quad \mathbf{d}\times\mathbf{b} = 0 \tag{4.2}$$

in the generic case (and to $\mathbf{d}\times\mathbf{b}=0$, $\Delta\eta= 0$ in the Bogomolny case). The first equation shows that, for solutions of (4.1), the generic finite-energy condition $\eta \to 0$ is sharpened to an exponential fall-off of η. (The BPS case escapes because $D^2\eta = 0$ is consistent with $\eta = b(\Omega)/r$).

Since $\Phi(\Omega)$ and $b(\Omega)$ are the only components of the field configuration that survive in the asymptotic region, within each topological sector defined by $\Phi(\Omega)$, the only possible asymptotic classification of the configurations is according to $b(\Omega)$. The conditions satisfied by $b(\Omega)$ are then contained in the second equation in (4.2), which may be written as

$$\mathbf{d}b \equiv \boldsymbol{\partial}b + [\mathbf{a}(\Omega), b] = 0\,. \tag{4.3}$$

This equation shows that $b(\Omega)$ is covariantly constant and thus lies on an H-orbit. Therefore $b(\Omega) = h_N(\Omega)Qh_N^{-1}(\Omega)$ in N and $b(\Omega) = h_S(\Omega)Qh_S^{-1}(\Omega)$ in S, where $Q = b(E)$ is in $\mathfrak{h}$. Plainly, Q is unique up to global gauge rotations, and there is thus no loss of generality in choosing it in a given Cartan algebra. In the singular gauge where $b(\Omega) = Q$, the loop (3.9) is simply

$$h(\varphi) = \exp 2iQ\varphi \qquad 0 \leqslant \varphi \leqslant 2\pi \tag{4.4}$$

and the periodicity of φ provides us with the <u>quantization condition</u>

$$\exp\ 4\pi i Q = 1\ . \tag{4.5}$$

Conversely, any quantized Q defines an asymptotic solution, namely

$$\mathbf{a} = \pm(1+\cos\theta)Q. \tag{4.6}$$

This shows that the solutions of the field equations can be classified asymptotically by the <u>charges</u> of H.

According to (3.8), the expression (3.8) for the "Higgs" quantum number m reduces to

$$m = 2\ \mathrm{tr}(Q\Psi)\ /\ \mathrm{tr}(\Psi^2)\ . \tag{4.7}$$

for those fields which are solutions of the field equations.

Let us now consider a monopole (given by) Q with Higgs charge m, and decompose it to central and semisimple parts $Q_{\shortparallel}$ and $Q_{\perp}$ respectively. According to (4.7) $2Q_{\shortparallel} = m\Psi$. Observe now that

$$z = \exp 4\pi Q_{\shortparallel} = \exp(-4\pi Q_{\perp}) \tag{4.8}$$

lies simultaneously in $Z(H)_o$, (the connected component of the centre of H) and in H_{ss}, and thus in $Z(H_{ss})$. Let us decompose $\mathfrak{h}_{ss}$ into simple factors, $\mathfrak{h}_{ss} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$, and denote by H_j the simple and simply connected group whose algebra is $\mathfrak{h}_j$. H_{ss} itself simply connected by assumption, so $Z(H_{ss}) = Z(H_1) \times \ldots \times Z(H_k)$ and thus $z = z_1 \ldots z_k$, where z_j is in $Z(H_j)$. However, as emphasised in Section 2, the central elements of a simple and simply connected group are in (1-1) correspondence with the minimal $\mathring{W}$'s and thus, for each z in the centre, there exists a unique set of $\mathring{W}_j$'s (where $\mathring{W}_j$ is either zero or a minimal vector of $\mathfrak{h}_j$) such that

$$z = (\exp -2\pi \mathring{W}_1)\ldots(\exp - 2\pi \mathring{W}_k) = \exp\Big\{-2\pi \sum_j^k \mathring{W}_j\Big\} = \exp -2\pi \mathring{W}^{(m)}. \tag{4.9}$$

$\mathring{W}^{(m)}$ depends only on the sector and not on Q itself, because all charges of a sector have the same Q_{11}. Hence the <u>entire sector</u> can be characterized by writing

$$2\mathring{Q}^{(m)} = m\Psi + \mathring{W}^{(m)} \tag{4.10}$$

By (4.8) $2\mathring{Q}^{(m)}$ is again a charge ($\exp 4\pi\mathring{Q}^{(m)} = 1$) and it obviously belongs to the sector m. Furthermore $\exp 4\pi(Q-\mathring{Q}) = \exp 4\pi Q \,.\, \exp(-4\pi\mathring{Q}) = 1$ shows that $2Q' = 2(Q-\mathring{Q})$ is in the charge lattice of H_{ss}. We conclude that any monopole is uniquely of the form

$$Q = \mathring{Q} + Q' = \mathring{Q} + \frac{1}{2}\sum_{i=1}^{r} n_i Q_i \,, \tag{4.11}$$

where the n_i are integers, and the Q_i, $i=1, \ldots r$ are the primitive charges of H_{ss}. The integers n_i could be regarded as secondary quantum numbers which supplement the Higgs charge m.

The expansion (4.11) will be crucial in our stability investigations. As a matter of fact, we shall show that $\mathring{Q}^{(m)}$ is the unique stable monopole in the sector m. The situation is conveniently illustrated on the diagrams of the examples in Section 7.

The classification of finite-energy solutions according to the secondary quantum numbers or, equivalently the matrix-valued charge Q, is convenient and illuminating, but in contrast to the classification of finite-energy configurations according to the Higgs charge m, it is not mandatory, in the sense that (for fixed m) the different charges Q are not separated by infinite energy barrriers. To see this one has only to note that the family of configurations [16]

$$\mathbf{A}_t = t\mathbf{A} + (1-t)\mathbf{A}' \ , \ \Phi^t = \Phi, \quad 0 \leqslant t \leqslant 1 \tag{4.12}$$

which are not in general solutions of (4.1) except for $t = 0,1$, but which interpolate smoothly between solutions $(\mathbf{A},\Phi)$ and $(\mathbf{A}',\Phi)$ and which lie in the same Higgs sector because Φ does not change, have finite energy for all $0 \leqslant t \leqslant 1$. Indeed, as $r \to \infty$ one has $\mathbf{A}^t \sim 1/r$, $r^{3/2}(D^t\Phi) = t(r^{3/2}D\Phi) + (1-t)(r^{3/2}D'\Phi) \to 0$, $r^3V(\Phi^t) = r^3V(\Phi) \to 0$, so that the integral (2.1) converges for $(\mathbf{A}^t,\Phi)$. As a matter of fact, one may obtain a rather simple and compact expression for the interpolated energy $E(\mathbf{A}^t)$ as follows: $D^t\Phi = t(D\Phi) + (1-t)(D'\Phi)$ and $B_{ij}{}^t = tB_{ij} + (1-t)B'_{ij} + t(1-t)\,[\Delta_i, \Delta_j]$, where $\Delta_i = \mathbf{A}_i - \mathbf{A}'_i$. This shows that the interpolated energy must be of the general form $E^t = at^2+b(1-t)^2+ct^2(1-t)^2+2ft(1-t)+2gt^2(1-t)+2ht(1-t)^2$, where $a,\ldots,g$ are integrals over the field configurations which are independent of t, and in particular $a = E$, $b = E'$ and $c = \int d^3x\ \mathrm{tr}[\Delta_i,\Delta_j]^2$, where E and E' are the energies of the solutions $(\mathbf{A},\Phi)$ and $(\mathbf{A}',\Phi)$. But since the solutions are extremal points of the energy, $\partial E^t/\partial t$ must vanish at $t= 0,1$ and this leads to the conditions $b = f+h$ and $a = f+g$. Using these two equations to eliminate h and g one finds that f is also eliminated and thus E^t reduces to the simple expression

$$E^t = t^2(3-2t)E + (1-t)^2(1+2t)\,E'^{\,2} + t^2(1-t)^2C/2. \qquad (4.13)$$

Since $\Delta \sim 1/r$ as $r \to \infty$ it is evident that C is finite, and hence that the interpolated energy is finite for all $0 \leqslant t \leqslant 1$. Thus the energy barrier between E and E' is finite. Note that in the special case that E = E' (4.13) reduces to $E^t = E + t^2(1-t)^2C$, which is just a standard quartic Higgs potential in t. But, of course, the interpolation (4.12) is not the optimal interpolation in this case since it does not, in general, follow the orbit of the gauge transformation (in H) which connects Q to Q' when E = E'.

5. Unstable solutions: reduction from an $\mathbb{R}^3$ to an S^2 problem.

In next two sections we wish to show that monopoles for which $Q' \neq 0$ are unstable with respect to variations of the gauge field and we have rather

$$\nu = 2 \sum_{q<0} 2|q|-1 \tag{5.1}$$

independent negative modes, where the sum goes over all negative eigenvalues q of Q (since $2Q$ is a charge, $2q$ is always an integer). We also wish to construct the negative modes explicitly. For these purposes it will be sufficient to consider only variations of the gauge potential of the Higgs group H for large values $r \geqslant R$ of the coordinates, and in this section we specify the variations more precisely and show that they effectively reduce the problem to the corresponding Yang-Mills problem on $\mathbf{S}^2$.

For H-valued variations of the gauge potentials alone, $\delta\Phi = 0$, $\delta\mathbf{A} = \mathbf{a} \in \mathfrak{h}$ say, the variations of the gauge field and covariant derivative are easily seen to be $\delta\mathbf{B} = \mathbf{D}\times\mathbf{a}$ $\delta^2\mathbf{B} = \mathbf{a}\times\mathbf{a}$ and $\delta(\mathbf{D}\Phi) = \mathbf{a}\Phi$. Note that all higher-order variations $\delta^3\mathbf{B}$ etc. are zero.

For the energy functional (3.1) the first variation is, of course, zero, since $(\mathbf{A},\Phi)$ is a solution of the field equations, and the higher-order variations are

$$\delta^2 E = \int d^3x \, \{\mathrm{tr}(\mathbf{D}\times\mathbf{a})^2 + \mathrm{tr}\,(\mathbf{B}\mathbf{a}\times\mathbf{a}) + (\mathbf{a}\Phi)^2\},$$

$$\delta^3 E = 2\int d^3x \, \mathrm{tr}\{(\mathbf{D}\times\mathbf{a})\mathbf{a}\times\mathbf{a}\}\,, \qquad \delta^4 E = 2\int d^3x \, \mathrm{tr}(\mathbf{a}\times\mathbf{a})^2 \tag{5.2}$$

all higher-order variations being zero. We shall assume that all variations are square-integrable, $(\mathbf{a},\mathbf{a}) = \int d^3x \mathrm{tr}(\mathbf{a})^2 < \infty$.

There are some general points worth noting. First, since $\delta\Phi = 0$, the only term in (5.2) that involves the Higgs field is $\mathrm{tr}(\mathbf{a}\Phi)^2$ and since $\mathbf{a}$ must be in the little group of $\Phi(\Omega) = \lim_{r\to\infty}\Phi(r,\Omega)$, in the case $V \neq 0$ this term vanishes asymptotically. Thus in the asymptotic limit $r \to \infty$ (5.2) becomes the variation of a pure Yang-Mills Lagrangian.

Second, the only term in (5.2) that involves radial derivatives is the $(\partial_r \mathbf{a})^2$ term in $(\mathbf{D}\times\mathbf{a})^2$ and this contribution may be written as

$$\delta^2 E_r = \int d^3x \, \mathrm{tr}(\partial_r \mathbf{a})^2 = \int dr d\Omega \, \mathrm{tr}(r\partial_r \mathbf{a})^2 =$$
$$= \int dr d\Omega \, \mathrm{tr}\,((d\,\mathbf{a})^2 + \mathbf{a}^2/4) = m^2 (\mathbf{a}, \mathbf{a}) \,, \qquad (5.3)$$

where d is the symmetrized dilatation operator $\{r,\partial_r\}/2$, δ^2 its average value and $m^2 \geq 1/4+\delta^2$. It can be shown (see Appendix) that the infimum of m^2 is $1/4$, and thus, although $\delta^2 E_r$ is not negligible because of the lower bound $1/4$, it can be reduced to this lower bound, and $\delta^2 E_r$ can be regarded as a mass term . Then the variations (5.2) are essentially variations on the 2-sphere S^2 for each value of r.

Finally, it should be noted that some of the variations, namely $\mathbf{a} = \mathbf{D}\chi + O(\chi^2)$, where χ is any scalar, are simply gauge transformations of the background field $\mathbf{A}$ and lead to zero energy variations. In particular, it is easy to verify that, because $\mathbf{A}$ satisfies the field equations, the second variation $\delta^2 E$ is zero for the infinitesimal variations $\mathbf{D}\chi$, and for this reason it is convenient to define the 'physical' variations $\mathbf{a}$ as those which are orthogonal to the $\mathbf{D}\chi$. Since χ is arbitrary, one has $\int d^3x \, \mathrm{tr}(\mathbf{a},\mathbf{d}\chi) = \int d^3x \, \mathrm{tr}(\mathbf{D}\mathbf{a},\chi) = 0 \Rightarrow$

$$\mathbf{D}.\mathbf{a} = 0 \qquad (5.4)$$

from which one sees that the physical variations may also be characterized as those which are divergence-free.

We shall be interested primarily in the second variation or <u>Hessian</u> $\delta^2 E$ in (5.2), and in finding the negative modes of this quantity. Since all the terms in $\delta^2 E$ are positive, except possibly the $\mathrm{tr}(\mathbf{B}\mathbf{a}\times\mathbf{a})$ term, negative modes are most likely to occur when this term is large and the others are as small as possible. Since $\mathbf{B}$ increases and the Higgs term decreases with r, this suggests that negative modes are most probable for large r, and hence we consider only variations whose support is asymptotic i.e.

$$\mathbf{a}(r,\Omega) = 0 \quad \text{for } r \leqslant R \tag{5.5}$$

where R is 'sufficiently large', and in practice this will mean R large enough for the asymptotic form of the fields (3.2) to be valid. We can then drop the Higgs terms in (5.2) and consider the pure Yang–Mills variations

$$\delta^2 E = \int d^3x \, \{ \, \mathrm{tr}(\mathbf{D}\times\mathbf{a})^2 + \mathrm{tr}\,(\mathbf{B}\mathbf{a}\times\mathbf{a})\}. \tag{5.6}$$

It will be convenient to write this expression in the form

$$\delta^2 E = \delta^2 E_1 + \delta^2 E_2 = \int d^3x \, \{\mathrm{tr}(\mathbf{D}\times\mathbf{a})^2 + \mathrm{tr}(\mathbf{D.a})^2\} + \int d^3x \, \mathrm{tr}\{(\mathbf{B}\mathbf{a}\times\mathbf{a} - (\mathbf{D.a})^2\}, \tag{5.7}$$

bearing in mind that $\mathbf{D.a}$ is unphysical and may be gauged to zero.

Let us first consider $\delta^2 E_1$. From the identity $(\mathbf{D}\times(\mathbf{D}\times\mathbf{a}))_i = \mathbf{D}^2 a_i - D_i(\mathbf{D.a}) - (\mathbf{B}\mathbf{a})_i$, where $(\mathbf{B}\mathbf{a})_i = [B_{ij}, a_j]$, one sees that

$$\delta^2 E_1 = \int d^3x \, \mathrm{tr}\,\{ - \mathbf{D}^2 + \mathbf{B}\mathbf{a}, \mathbf{a}\} = \delta^2 E_r + \int d^3x \, r^{-2} \, \mathrm{tr}(\mathbf{L}^2 + \mathbf{b}\mathbf{a}, \mathbf{a}), \tag{5.8}$$

where $\mathbf{L} = \mathbf{x} \times \mathbf{D}$ and $\delta^2 E_r$ is the radial part discussed above.

It is well-known that the components of $\mathbf{L}$ do not satisfy the angular momentum relation, $[L_i, L_j] = \epsilon_{ijk} \, [L_k + x_k(\mathbf{x.B})] \neq \epsilon_{ijk} \, L_k$, but that for spherically symmetric, and hence for asymptotic, fields, the quantities $\mathbf{J}$ obtained by adding $\mathbf{x}.(\mathbf{x.B})$ to $\mathbf{L}$ do satisfy such an algebra i.e.

$$[J_i, J_k] = \epsilon_{ijk} J_k \, , \qquad \text{where} \qquad J_i = L_i + x_i(\mathbf{x.B}) \tag{5.9}$$

Furthermore, since $\mathbf{x}$ and $\mathbf{L}$ are orthogonal, we have $\mathbf{J}^2 = \mathbf{L}^2 + b^2$. Hence we may write

$$\delta^2 E_1 = \int dr d\Omega \; \mathrm{tr}\{(J^2 - b(b-1))\mathbf{a},\mathbf{a}\} + \delta^2 E_r . \qquad (5.10)$$

Now, since $\mathbf{J}$ also commutes with $\mathbf{x.B}$ in the asymptotic region, it is convenient to decompose the variations $\mathbf{a}$ into eigenmodes of $\mathbf{B}$ i.e. to write

$$(1/r^2)\,[B_{ij}, a_j] = q\, a_i \qquad (5.11)$$

where the q 's are the eigenvalues. This is possible and the q 's will be real because B_{ij} is skew-symmetric in the Lie algebra as well as in the vector space, and indeed, because of this, it is easy to see that the q 's come in pairs of opposite sign and multiplicity two i.e. in quadruplets $(q,q,-q,-q)$. As a matter of fact, the eigenvalues are $q = q_\alpha$ where the q_α are the charges defined in (2.1) and have the two-dimensional eigenspaces spanned by X_α and Y_α. Then on each q -sector $\delta^2 E_1$ will be

$$\delta^2 E_1 = \int dr d\Omega \; \mathrm{tr}\left((m^2 + \mathcal{C} - q(q-1))\mathbf{a},\mathbf{a}\right) \qquad (5.12)$$

where $\mathbf{a}$ is now parallel to X_α or Y_α and $\mathcal{C} = j(j+1)$ $(j \geqslant 0)$ is the <u>Casimir</u> of the angular momentum algebra generated by $\mathbf{J}$. Note that j must be integer or half-integer, according as q is integer or half-integer, because q is the only non-orbital contribution to $\mathbf{J}$. Now since $\delta^2 E_1$ in (5.7) is manifestly positive we must have

$$m^2 + \mathcal{C} - q(q-1) = (1/4 + \delta^2) + j(j+1) - q(q-1) \geqslant 0 \qquad (5.13)$$

and since δ^2 is arbitrarily small, we see that $j \geqslant |q|-1$. Note that $j \geqslant |q|-1$ follows from the manifest positivity of δE_1 and does not require any special pleading, such as that used, for example, by Coleman [2]. It follows from (5.13) that the only possible eigenvalues of j are $|q|-1$, $|q|$, $|q|+1,\ldots$ .In particular the value $j = |q|-1$ can occur only for $q \leqslant -1$. It will be convenient to separate the $j = |q|-1$ and $j \geqslant |q|$ cases and to write

$$\delta^2 E_1 = m^2\,(\mathbf{a},\mathbf{a}) \qquad \text{for } j = |q|-1, \quad |q| \geqslant 1 \tag{5.14}$$

and

$$\delta^2 E_1 = \{m^2 + (j-q)(j+q+1)\}\;(\mathbf{a},\mathbf{a}) \qquad \text{for } j \geqslant |q|. \tag{5.15}$$

Let us now consider $\delta^2 E_2$. Using the equality $\mathrm{tr}(\mathbf{B}\mathbf{a}\mathrm{x}\mathbf{a}) = \mathrm{tr}(\mathbf{a}\mathbf{B}\mathbf{a}) = r^{-2}\mathrm{tr}(\mathbf{a}\mathbf{b}\mathbf{a}) = q\; r^{-2}\mathrm{tr}(\mathbf{a},\mathbf{a})$, and the fact that $\mathbf{D}.\mathbf{a}$ is zero on the physical states, we see that

$$\delta^2 E_2 = q\,(\mathbf{a},\mathbf{a}) \tag{5.16}$$

From the positivity of $\delta^2 E_1$ we then see that the Hessian $\delta^2 E$ will be positive unless q is negative. Furthermore, when q is negative, (5.15) becomes

$$\delta^2 E_1 = \{m^2 + (j+|q|)(j-|q|+1\}\,(\mathbf{a},\mathbf{a}) \;\geqslant 2|q|\,(\mathbf{a},\mathbf{a}) \tag{5.17}$$

so, for $j \geqslant |q|$, the restriction of $\delta^2 E_1$ to the physical states will dominate $\delta^2 E_2$ and the Hessian will again be positive. It follows that the only possibility for negative modes of the Hessian is when $q \leqslant -1$ and $j = |q|-1$, in which case,

$$\delta^2 E \;= (m^2 - |q|)\,(\mathbf{a},\mathbf{a}) < 0 \tag{5.18}$$

Note that since in the case $j = |q|-1$ the first term on the r.h.s. of (5.7) vanishes, the variation actually satisfies the simple equations

$$\mathbf{d}\times\mathbf{a} = 0, \qquad \mathbf{d}.\mathbf{a} = 0 \tag{5.19}$$

where $\mathbf{d} = r\mathbf{D}$. In particular the equation $\mathbf{D}.\mathbf{a} = 0$ shows that they are true physical modes. It is also easy to see that these modes form a $2j+1 = 2|q|-1$ dimensional multiplet of the $\mathbf{J}$ – algebra. We shall see in the next section that for each $|q|$ there is one and only one such multiplet. Thus finally we have the result that the monopole is unstable if, and only if, $|q| \geq 1$ and that it has $2\,(2|q|-1)$ negative modes (corresponding to the two negative q's in the quadruplet $(q,q,-q,-q)$. This proves the index formula (5.1).

The simplest way of counting the number of instabilities for $j \geq |q|$ is to use the diagram (see examples in Section 7) ; (1.2) is twice the number of times the straight line drawn from the origin to $2Q$ intersects the root planes. The opposite condition $|q| \leq 1/2$ is, of course, just the Brandt-Neri stability condition [1,2,4] . From the discussion of Section 4 we know however that $|q| \leq 1/2$ if and only if $Q = \mathring{Q}$ is a sum of minimal $\mathring{W}$'s i.e. when Q' in eqn. (4.11) is zero: $\mathring{Q}$, given by (4.10), is the unique stable monopole of the topological sector under consideration.

For BPS monopoles the above argument breaks down: due to the b/r term in the expansion (3.6) of the Higgs field, the second variation picks up an extra term $\mathrm{tr}([\mathbf{b},\mathbf{a}])^2$ and we get rather

$$\delta^2 E = \delta^2 E_r + \int dr d\Omega \, \{\mathrm{tr}(\mathbf{J}^2 - b(b-1)\mathbf{a}, \mathbf{a}) + \mathrm{tr}([\mathbf{b},\mathbf{a}])^2\} \tag{5.20}$$

On the q-eigenspace of b the new term is q^2, which just cancels the $-q^2$ in eqn. (5.10), and consequently the total Hessian is manifestly positive,

$$\delta^2 E = ((m^2 + \mathbf{J}^2)\mathbf{a}, \mathbf{a}) > 0. \tag{5.21}$$

BPS monopoles are hence stable under variations of the gauge field alone, even if their charge is not of the form (4.10). In particular, a BPS monopole whose non-Abelian charge is a non-minimal primitive W is stable [17] (while for $V \neq 0$ it would be unstable).

In conclusion it might be worth remarking that the instability index $(\alpha|q|-1)$ can be related to the Atiyah-Singer index for the Dirac operator and to the Witten index for supersymmetry. Indeed if one writes the negative mode equations (5.19) in their complex form,

$$\mathbf{Da} = 0 \quad \text{where} \quad D = D_1 + D_2 \quad , \quad a = a_1 - iq_2 \,, \tag{5.22}$$

one sees at once that they are Dirac-like equations and hence the number of solutions is an AS-index. (Note however, that since **a** is supposed to be a 2-vector, whereas the conventional AS-equation is for a 2-spinor, there is an extra factor $g^{1/4}$ in the inner product for **a**, and this results in a reduction from $2|q|$ to $2|q|-1$ solutions. In other words the instability index is the AS-index for <u>vectors</u>).

To obtain the relation to the Witten index one needs only note that the Hamiltonian

$$H = \begin{bmatrix} 0 & D_1 + iD_2 \\ D_1 - iD_2 & 0 \end{bmatrix} \tag{5.23}$$

is supersymmetric, and that (5.22) is just the equation that defines the ground ($H = 0$) states of this Hamiltonian.

6. Negative modes

In this section we wish to show that the negative modes anticipated in Section 5 actually exist and to construct them explicitly. For this purpose it is convenient to use stereographic coordinates on S^2 i.e. to use 3-dimensional coordinates (r,z), where $z = x + iy = e^{i\varphi} \tan\theta/2$ and r,θ,φ are the usual polar coordinates. In the (r,z) coordinates the 3-dimensional line element and the 2-dimensional surface element become, respectively, $ds^2 = dr^2 + (r/\rho)^2(dx^2+dy^2)$ and $r^2 d\Omega = (r/\rho)^2\, dxdy$, where $\rho = 1+x^2+y^2 = 1+z\bar{z}$. The advantage of the stereographic coordinates is that they are the simplest case of conformal coordinates, for which one has the special relation $\sqrt{g}\, g^{\alpha\beta} = \delta_{\alpha\beta}$, $(\sqrt{g} = \sqrt{\det g_{\alpha\beta}})$, where $g_{\alpha\beta}$ is the restriction of the metric tensor to S^2. In particular the inner product of two vectors on S^2 becomes

$$(a,b) = \int dxdy \sqrt{g}\, g^{\alpha\beta} a_\alpha b_\beta = \int dxdy\, (a_\alpha, b_\alpha) \tag{6.1}$$

$(-\infty \leqslant x, y \leqslant \infty)$, and thus it reduces to the usual planar inner-product for functions of x and y. It is easy to see that in the stereographic coordinate the background gauge-potential $A_\varphi = Q(1-\cos\theta)$ becomes

$$A_x = Q\, y/\rho, \qquad A_y = -\,Q\, x/\rho \tag{6.2}$$

Let us now consider the part $\delta^2 E_1$ of the Hessian, which played a central role in Section 5. From eqn. (5.7) one may write $\delta^2 E_1 = \int d^3x\, (\partial_r \mathbf{a}) + \int r^2 dr K$, where, in stereographic coordinates,

$$K = \int dxdy\, g^{1/2}\ \mathrm{tr}\{(\epsilon^{\alpha\beta} D_\alpha a_\beta)^2 + (g^{-1/2} D_\alpha \sqrt{g}\, g^{\alpha\beta} a_\beta)^2\} =$$

$$= \int dxdy\, g^{-1/2}\, \mathrm{tr}\{(D_1 a_2 - D_2 a_1)^2 + (D_1 a_1 + D_2 a_2)^2\} =$$

$$= \int dxdy\, \rho^2\ \mathrm{tr}\,(D_z a)^2 , \tag{6.3}$$

where $D_z = D_1 + iD_2$, $a = a_1 - ia_2$. On account of the form (6.2) for the gauge-potential it is natural to absorb the ρ^2 in D_z by defining

$$\Delta(Q) = i\rho D_z = i(\rho\partial_z + \bar{z}Q) \quad (\text{so that } \Delta^+(Q) = i(\rho\partial_{\bar{z}} + Qz)) \tag{6.4}$$

and then (using partial integration) K may be rewritten in the more compact form

$$K = \int dxdy\, \mathrm{tr}(a, \Delta^+(Q)\Delta(Q)a)\,. \tag{6.5}$$

We now wish to relate the operator $\Delta^+(Q)\Delta(Q)$ in (6.4) to the total angular momentum operator J^2 of Section 5. First, we note that in stereographic coordinates the components of J are given by

$$J_+ = i(\bar{\partial} + z^2\partial + Qz), \quad J_- = i(\partial + \bar{\partial}\bar{z}^2 - zQ), \quad J_3 = z\partial - \bar{z}\bar{\partial} + Q - 1 \tag{6.6}$$

where ∂ means ∂_z and $\bar{\partial}$ means $\partial_{\bar{z}}$. Indeed, it is easy to verify directly that the operators in (6.6) satisfy the so(3) algebra. On the other hand, by using the identity

$$z^2\partial = zM + (z\bar{z})\bar{\partial}\,, \quad \text{where} \quad M = z\partial - \bar{z}\bar{\partial}, \tag{6.7}$$

one sees that the operators $J_\pm$ in (6.6) may also be written as

$$J_+ = i(\rho\bar{\partial} + z(1+J_3))\,, \quad J_- = i(\rho\partial + \bar{z}(1-J_3)) \tag{6.8}$$

which shows that the $J_\pm$ are operators of the same kind as occur in (6.4) but with Q replaced by $R = 1-J_3$. In other words,

$$J_+ = \Delta^+(R), \quad J_- = \Delta(R) \text{ and } J^2 = \Delta^+(R)\Delta(R) + J_3(J_3-1) \tag{6.9}$$

Hence to relate $\Delta^+(Q)\Delta(Q)$ to $\mathbf{J}^2$ we need only relate it to $\Delta^+(R)\,\Delta(R)$. Now a straightforward computation shows that

$$\Delta^+(Q)\Delta(Q) \;=\; \Delta^+(0)\,\Delta(0) - Q(Q-1) + \rho Q(Q+M-2) \qquad (6.10)$$

and since in the computation the only property of Q that is used is that Q commutes with $z\bar{z}$ and $\bar{z}\bar{\partial}$, and this is true also of R, a similar formula holds for R. Furthermore the last term in (6.10) (coefficient of ρ) is invariant under the interchange $Q \leftrightarrow R$. It follows that

$$\Delta^+(Q)\Delta(Q) + Q(Q-1) \;=\; \Delta^+(R)\Delta(R) + J_3(J_3-1) \qquad (6.11)$$

and since the right-hand-side of eqn. (6.11) is just the Casimir $\mathbf{J}^2$, one sees that the required relation between $\Delta^+(Q)\Delta(Q)$ and J^2 is

$$\Delta^+(Q)\Delta(Q) \;=\; \mathbf{J}^2 - Q(Q-1), \qquad (6.12)$$

in agreement with the results of Section 5.

Up to this point, of course, we have only reproduced the results of Section 5 in stereographic coordinates. The point, however, is that in these coordinates it is easy to construct the negative modes (indeed the whole eigenspace of the Hessian) explicitly.

Let us first consider the negative modes. As already discussed in Section 5, for each eigenspace spanned by X_α, Y_α such that $q = q_\alpha \leqslant -1$, the negative modes are solutions to the two coupled equations in (5.19). Adding (−i) – times the first term to the second we get

$$D_z a = (\rho\partial + \bar{z}q)a = 0, \qquad (6.13)$$

whose general solution is

$$a^{(k)} = \bar{z}^{k}/\varrho^{|q|} = \bar{z}^{k}/(1+z\bar{z})^{|q|}, \qquad k = 0,1,... \tag{6.14}$$

From the expression (6.1) for the inner product one sees that the $a^{(k)}$are square-integrable if and only if $k \leqslant 2|q|-2$. We conclude that the negative modes are just the $a^{(k)}$'s in eqn. (6.14) for which $\underline{0 \leqslant k \leqslant 2|q|-2}$, and there are just $2|q|-1$ (i.e. a $2|q|-1$-dimensional multiplet) of them, as anticipated. In polar coordinates θ,φ these negative modes are also expressed as

$$a_\theta = (1/2)\, e^{-i(k+1)\varphi} (\sin\theta/2)^k (\cos\theta/2)^{2q-2-k}\, X_\alpha,$$

$$a_\varphi = e^{-i(k+1)\varphi} (\sin\theta/2)^{k+1} (\cos\theta/2)^{2q-1-k} X_\alpha, \quad 0 \leqslant k \leqslant 2|q|-2. \tag{6.15}$$

and with X_α replaced by Y_α respectively.

This result can also be understood in a geometric framework [5]: the $a^{(k)}$'s are antiholomorphic sections of suitable line bundles. This is not a coincidence, since these holomorphic sections of line bundles are exactly the representation spaces of the rotation group SU(2)).

Now we turn to the remaining eigenspace of the Hessian. From (5.7) and (5.10) one sees that they are just the eigenspaces of $\mathbf{J}^2$ modulo zero modes. Hence it suffices to consider the eigenspace of $\mathbf{J}^2$ i.e. the weights of the various representations of $\mathbf{J}^2$. Furthermore, since any weight can be obtained from the lowest (or highest) weight in a given irreducible representation $\mathbf{J}^2 = j(j+1)$ by the repeated application of $J_\pm$, it suffices to consider the lowest (or highest) weights. These are evidently defined as those a's for which

$$J_- a = 0, \quad J_3 a = -j a \quad (\text{or } J_+ a = 0 \text{ and } J_3 a = ja) \tag{6.16}$$

Since for a given j the operators $J_{\pm}$ in (6.8) become $i(\varrho\bar{\partial} + zj(j+1))$ and its hermitian conjugate, eqns. (6.16) may be written as

$$(\varrho\bar{\partial} + z(J+1))\, a = 0, \quad (z\partial - \bar{z}\bar{\partial})\, a = (-j+q+1)\, a \tag{6.17}$$

and its conjugate. The solution of (6.17) is easily seen to be unique and of the form

$$a = z^{-j-|q|+1} / \varrho^{j+1} = z^{-j-|q|+1} / (1+z\bar{z})^{|q|} \tag{6.18}$$

Thus there is just one multiplet for each j and its highest and lowest weights are just (6.18) and its conjugate respectively. The negative mode multiplet is just that for $j = |q|-1$, and it might be worth remarking that, in contrast to this case, the weights for $j \geqslant |q|$ do no satisfy the gauge condition $\mathbf{D}.\mathbf{a} = Da + Da = 0$, and hence are mixtures of physical and gauge (zero-mode) states.

7. Examples

The simplest case of interest is that of when the little group H of the Higgs field is H = U(2). The Cartan algebra of u(2) is two-dimensional, it is in fact the set of all diagonal hermitian matrices. The vertical axis on Fig. 1 represents the centre generated by diag (1,1), and the horizontal axis t' consists of multiples of σ_3= diag (1,−1). The only positive root is $\alpha(X) = X_1 - X_2$, the difference of the diagonal entries. The corresponding primitive vector, $\mathring{W} = \sigma_3/2$ = diag (1/2,−1/2) is also a minimal one. In fact, $\exp 2\pi i \mathring{W} = -1$.

The vertical lines intersecting t' in integer multiples of $\mathring{W}$ are the root planes. $Q_1 = 2\mathring{W} = \sigma_3$ generates the charge lattice of H_{ss} = SU(2) which is also the topological zero-sector of U(2). Those charges on the same horizontal lines form the topological sectors labelled by a single integer m, defined by $2Q_{11}$ = m diag (1/2,−1/2) = m Ψ. Sector 1 is shifted with respect to the 0-sector by $\Psi + \mathring{W}$. Remarkably, all even (respectively odd) sectors reproduce the same pattern.

The unique stable monopole of the sector m is

$$2\mathring{Q}^{(m)} = m\Psi + \mathring{W}_{[m]} = \begin{cases} \mathrm{diag}\,(k,k) & \text{for } m = 2k \\ \mathrm{diag}\,(k+1,\,k) & \text{for } m = 2k+1 \end{cases} \tag{7.1}$$

where [m] is m modulo 2 and $\mathring{W}_0 = 0$ by convention. Any other monopole of the sector m is of the form

$$2Q^{(m)} = 2\mathring{Q}^{(m)} + nQ_1 = 2\mathring{Q}^{(m)} + \mathrm{diag}\,(n,-n). \tag{7.2}$$

Those monopoles for which $n \neq 0$ are unstable, with index $\nu = 2(2n-1)$ for m even and $\nu = 4n$ for m odd. For example, when G = SU(3) is broken to U(2) by an adjoint Higgs Φ, the vacuum sector contains a configuration (whose existence has been rigorously proved in [18]) whose non-Abelian charge Q is conjugate to diag (1/2, -1/2, 0) [19]. This configuration is indeed unstable and has 2 negative modes, namely

$$a_\theta = (1/2)\, e^{-i\varphi}\, \sigma_i, \qquad a_\varphi = e^{-i\varphi} \sin\theta\, \sigma_i, \qquad (i = 1,2) \tag{7.3}$$

where σ_i are the off-diagonal Pauli matrices in $su(2) \subset u(2)$.

The physically most relevant example (at present) is when the Higgs little group is $\underline{H = U(3)}$ (locally $su(3)_c + u(1)_{em}$), which may well be the exact symmetry group in nature of the strong and electromagnetic interactions.

The diagram is now three-dimensional, with the central u(1) being the vertical axis on Fig. 2 and t' the horizontal plane shown in Fig. 3, which is in fact the diagram of SU(3). The primitive roots are $\alpha_1(X) = X_1 - X_2$ and $\alpha_2(X) = X_2 - X_3$ for $X = \mathrm{diag}\,(X_1, X_2, X_3)$. The corresponding primitive vectors are $\mathring{W}_1 = \mathrm{diag}\,(2/3, -1/3, -1/3)$ and $\mathring{W}_2 = \mathrm{diag}\,(1/3, 1/3, -2/3)$. They are also minimal vectors; their exponentials are in bijection with the elements in the $\mathbb{Z}_3$-centre of SU(3).

The three families of root planes correspond to the positive roots α_1, α_2 and the highest root $\theta = \alpha_1 + \alpha_2$. The charge lattice of $H_{ss} = SU(3)$ is generated by $Q_1 = \mathrm{diag}\,(1,-1,0)$ and $Q_2 = \mathrm{diag}\,(0,1,-1)$.

Those charges lying in the same horizontal plane form the topological sectors, labelled by an integer m. In fact, the projection of an entire sector onto the centre of u(3) is $m\Psi = m\ \mathrm{diag}\,(1/3, 1/3, 1/3)$. The unique stable monopole in the sector m is

$$Q(m) = m\Psi + W_{[m]} = \begin{cases} \mathrm{diag}\,(k,k,k) & \\ \mathrm{diag}(k+1,k,k) & \text{for } m = 3k+1 \\ \mathrm{diag}(k+1,k+1,k) & \text{for } m = 3k+2 \end{cases} \tag{7.4}$$

where [m] means m modulo 3. All other monopoles are

$$2Q = 2\overset{\circ}{Q} + n_1 Q_1 + n_2 Q_2 = 2\overset{\circ}{Q} + \operatorname{diag}(n_1, n_2 - n_1, -n_2) . \tag{7.5}$$

and are thus unstable for n_1 or n_2 not equal to zero. Sector 1 (respectively Sector 2) are obtained form the 0-sector by adding Ψ and shifting by $\overset{\circ}{W}_1$ and $\overset{\circ}{W}_2$ respectively, and the pattern is periodic modulo 3 .

Those configurations for which $Q' \neq 0$ are unstable. The simplest way of counting the index is by using the diagram . For example, the monopole whose charge is $2Q = \operatorname{diag}(2,0,-2)$ belongs to the vacuum-sector (since its charge is in $H_{ss} = SU(3)$) . Furthermore $\alpha_1(2Q) = 2$, $\alpha_2(2Q) = 2$ and $\theta(2Q) = 4$, and so the index is 10. The negative modes are given by (6.15).

To have a simple example where not all primitive weights are minimal, let us assume, that the residual group is $H = \underline{(U(1) \times Sp(4))/\mathbb{Z}_2}$. Then $\mathfrak{h}_{ss} = sp(4) = so(5)$ and H_{ss} is Spin(5), the double covering of SO(5). $\mathfrak{h}_{ss}$ can be represented by 4x4 symplectic matrices with a 2-dimensional Cartan algebra, say $\mathfrak{t}' = \operatorname{diag}(a,b,-a,-b)$. The charge lattice consists of vectors in $\mathfrak{t}'$ with integer entries. Let us choose the primitive roots $\alpha_1 = \operatorname{tr}(H_1 .)$ and $\alpha_2 = \operatorname{tr}(H_2 .)$, where

$$H_1 = (1/2) \operatorname{diag}(1,-1,-1,1) \quad \text{and} \quad H_2 = (1/2) \operatorname{diag}(0,1,0,-1) \tag{7.6}$$

Those vectors dual to the primitive roots are

$$W_1 = \operatorname{diag}(1,0,-1,0) \quad \text{and} \quad \overset{\circ}{W}_2 = (1/2) \operatorname{diag}(1,1,-1,-1) \tag{7.7}$$

Any of the properties a), b), or c) of Section 2 shows that only $\overset{\circ}{W}_2$ is minimal: For example, only $\overset{\circ}{W}_2$ exponentiates into the non-trivial element (-1) of Sp(4): $\exp 2\pi W_1 = 1$, $\exp 2\pi W_2 = -1$. In other words, while W_1 is already a charge, $\mathring{W}_2$ is only half-of-a charge (Fig. 4). Alternatively, the two remaining positive roots are $\varphi = \alpha_1 + \alpha_2$ and the highest root $\theta = 2\alpha_1 + \alpha_2$. Thus there are hence 4 families of root - planes.

Let the integer m label the topological sectors. For m even, m = 2k, the unique stable monopole belongs to the centre,

$$2\mathring{Q}^{(2k)} = 2k\ \Psi, \tag{7.8}$$

where Ψ is a generator of the centre normalizd so that 2Ψ is a charge. For m odd, m = 2k+1, the unique stable monopole is rather

$$2\mathring{Q}^{(2k+1)} = (2K+1)\ \Psi + \mathring{W}_2. \tag{7.9}$$

It may be worth noting that, in contrast to the H_{ss} = SU(N) case, $2Q = W_1$ is an ustable monopole in the vacuum sector, which has index $2\ (\theta(W_1) - 1) = 2$. (Remark, that if W_1 was the charge of a Prasad-Sommerfield monopole, it would be stable [17]).

The negative modes are expressed once more by (7.3), but this time $\sigma_\pm$ means rather

$$\sigma_+ = (1/2)\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{bmatrix} \qquad \sigma_- = (1/2)\begin{bmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & i \\ & & -i & 0 \end{bmatrix} \tag{7.10}$$

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Appendix

PROPOSITION: $m^2(\mathbf{a},\mathbf{a}) = \int d\Omega dr\ \mathrm{tr}(r\,\partial_r \mathbf{a})^2 \geq ((1/4) + \delta^2)\ (\mathbf{a},\mathbf{a})$

and $\inf m^2 = 1/4$.

Proof:

$$\int_0^\infty dr\,\mathrm{tr}\left(r^2(\partial_r \mathbf{a})^2\right) = \int_R^\infty \mathrm{tr}(r\partial_r \mathbf{a} + \mathbf{a}/2)^2\,dr - \int_R^\infty \mathrm{tr}\,(r\mathbf{a}\partial_r \mathbf{a})dr - (1/4)\int_R^\infty dr\ \mathrm{tr}\,(\mathbf{a}^2)$$

$$= \int_R^\infty \mathrm{tr}(r\partial_r \mathbf{a} + \mathbf{a}/2)^2 + (1/4)\int_R^\infty dr\,\mathrm{tr}\,\mathbf{a}^2 + \mathrm{tr}\,\mathbf{a}^2(R)\ R\ /2\ .$$

Therefore, $m^2 \geq \frac{1}{4} + \delta^2$.

Equality can never be achieved because $r\partial_r\mathbf{a} + \mathbf{a}/2 = 0 \Rightarrow \mathbf{a}$ is proportional to $r^{-1/2} \Rightarrow R\mathbf{a}^2(R) \neq 0$. However, consider $\mathbf{a} = f(r)\ \beta(\Omega)$ where $\beta(\Omega)$ is a vector on S^2, and

$$f(r) = \begin{cases} (r-R)/R & R \leqslant r \leqslant 2R \\ 2R/r)^{1/2} & 2R \leqslant r \leqslant 2sR \\ s^{1/2}R & 2sR \leqslant r \end{cases}$$

Then $\int (r\partial_r f)^2 dr / \int f^2 dr = (17 + 3\ln s)\ (5 + 12\ln s) \to 1/4$ as $s \to \infty$, showing that $1/4$ is indeed the infimum.

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Figure caption

Figure 1. Diagram of U(2). The horizontal axis represents the Cartan algebra (multiples of σ_3 = diag(1,-1)) of the semi-simple part h_{ss} = su(2), and the vertical axis represents the centre generated by Ψ = (1/2) diag (1,1). The only root is tr(σ_3.). The charges are $Q = m\Psi + \mathring{W}_{[m]} + n\sigma_3$, where m is the topological quantum number [m] means m modulo 2 and n is a "secondary" quantum number. $\mathring{W} = \sigma_3/2$ is a minimal vector, dual to the root. The root planes are vertical lines which intersect the horizontal axis at integer multiples of W. The horizontal lines are the topological sectors (labeled by m). In each sector, the charge which is the closes to the centre is the unique stable monopole there. For other charges the number of instabilities is twice the intersection with the root planes. Sector 1 is obtained from the vacuum sector by shifting by $\mathring{W}$. The pattern is periodic in m modulo 2.

Figure 2. Diagram of U(3). The vertical axis represents the centre generated by Ψ = (1/3) diag (1,1,1), and the horizontal plane t' is the Cartan algebra of SU(3) shown in more detail on Fig. 3. The charges are $m\Psi+\mathring{W}_{[m]} + n_1Q_1+n_2Q_2$, where [m] means m modulo 3, n_1 and n_2 are integers and Q_1, Q_2 are the primitive charges of SU(3) on Figure 3. The horizontal planes are the topological sectors. Sector m is obtained from the vacuum-sector by shifting by $\mathring{W}_{[m]}$. In each sector, the charge closest to the centre is that of the unique stable monopole and the number of instabilities is twice the intersection with the root planes. The diagram is periodic in m modulo 3.

Figure 3. Diagram of SU(3). Q_1 = diag (1,-1,0) and Q_2 =diag (0,1,-1), are the primitive charges and the two primitive roots roots are tr(Q_1 .) and tr(Q_2 .) . The minimal vectors $\mathring{W}_1$ = (1/3) diag (2,-1,-1) and $\mathring{W}_2$ = (1/3) diag (1,1,-3) generate the diagram. There are three root planes, intersecting in angle $\pi/3$.

Figure 4 The diagram of Sp(4) Spin(5), the double-covering of SO(5). The primitive charges are Q_1 = diag(1,0,-1,0) and Q_2 = diag(0,1,0,-1). The two primitive W 's are W_1 = diag (1,0,-1,0) and $\mathring{W}_2$ = (1/2) diag (1,1,-1,-1), out of which only $\mathring{W}_2$ is minimal.

TABLE CAPTION

Table 1.

The simply connected simple compact Lie groups with non - trivial centres, their minimal co - weights (expressed as matrices and as primitive weights), the representations characterized by co - weights, and the expansions of the highest roots in terms of primitive roots. Here σ_2, σ_3 denote Pauli matrices, γ_μ Clifford matrices, y the SU(3) hypercharge diag(2,-1,-1) and $\gamma = \gamma_1\gamma_2 \cdots \gamma_{4n}$.

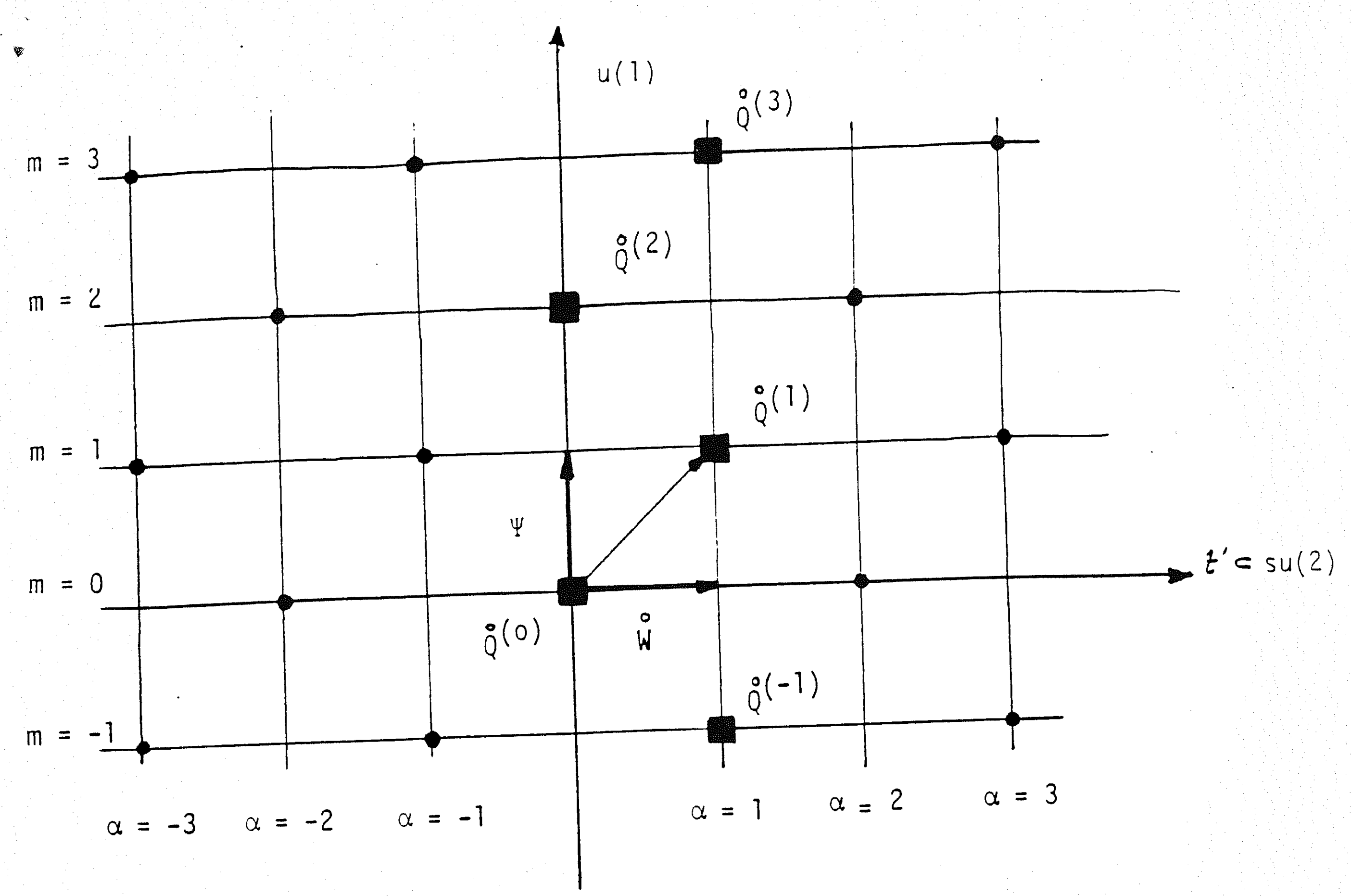


Figure 1.

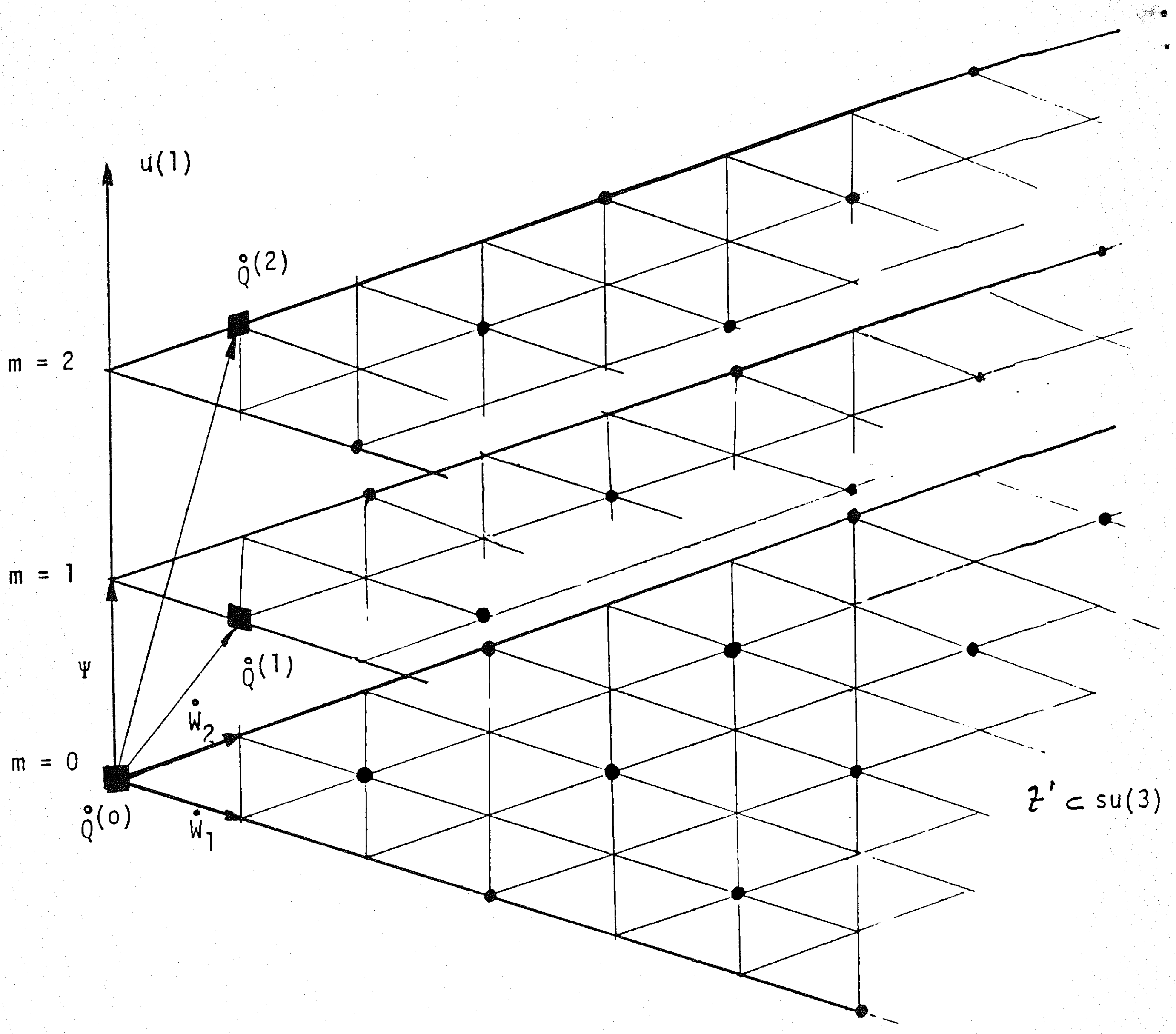

u(1)
$\mathring{Q}^{(2)}$
m = 2
m = 1
Ψ
$\mathring{Q}^{(1)}$
$\mathring{W}_2$
m = 0
$\mathring{Q}^{(o)}$
$\mathring{W}_1$
$\mathcal{Z}' \subset su(3)$

Figure 2.

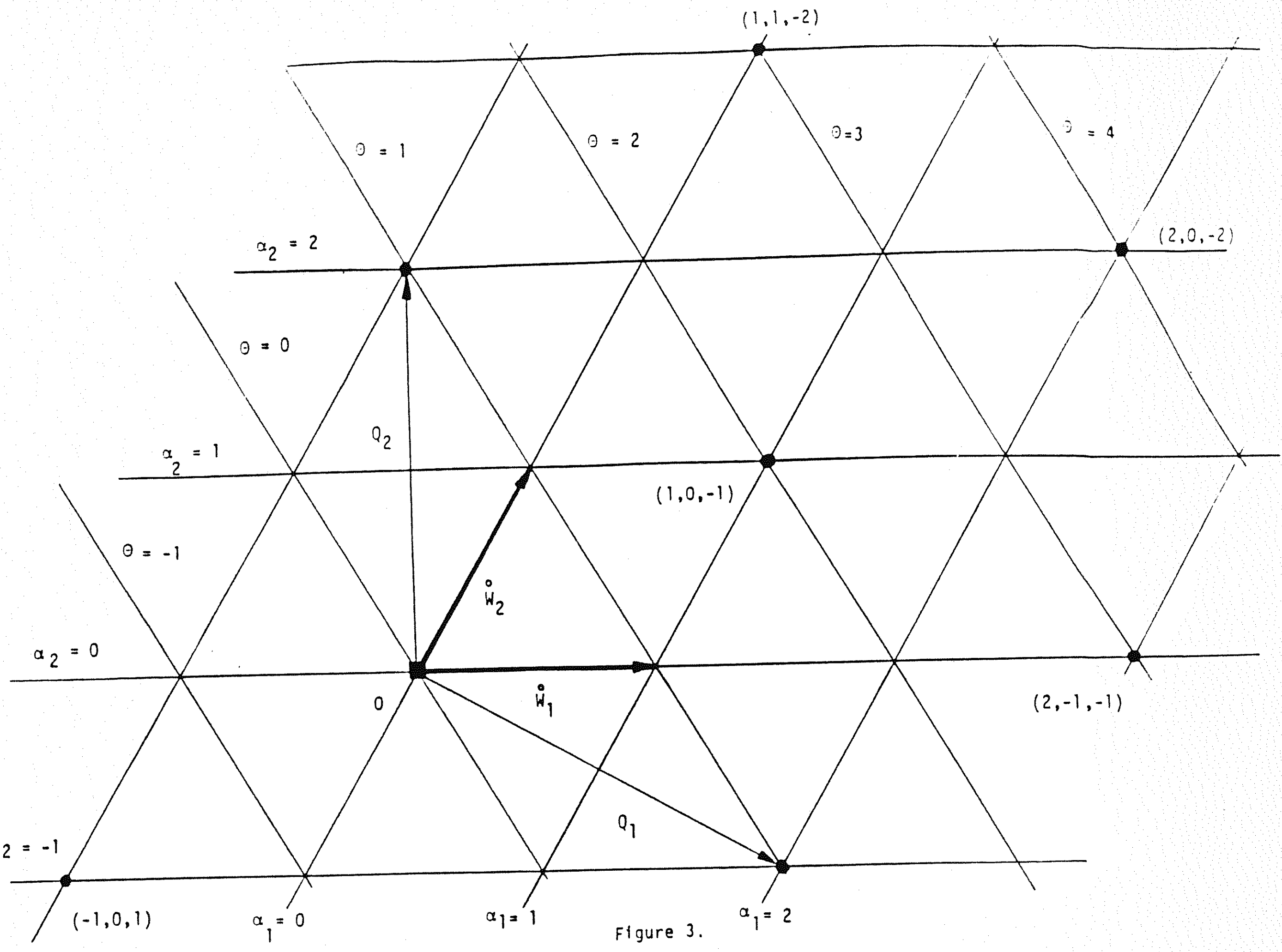
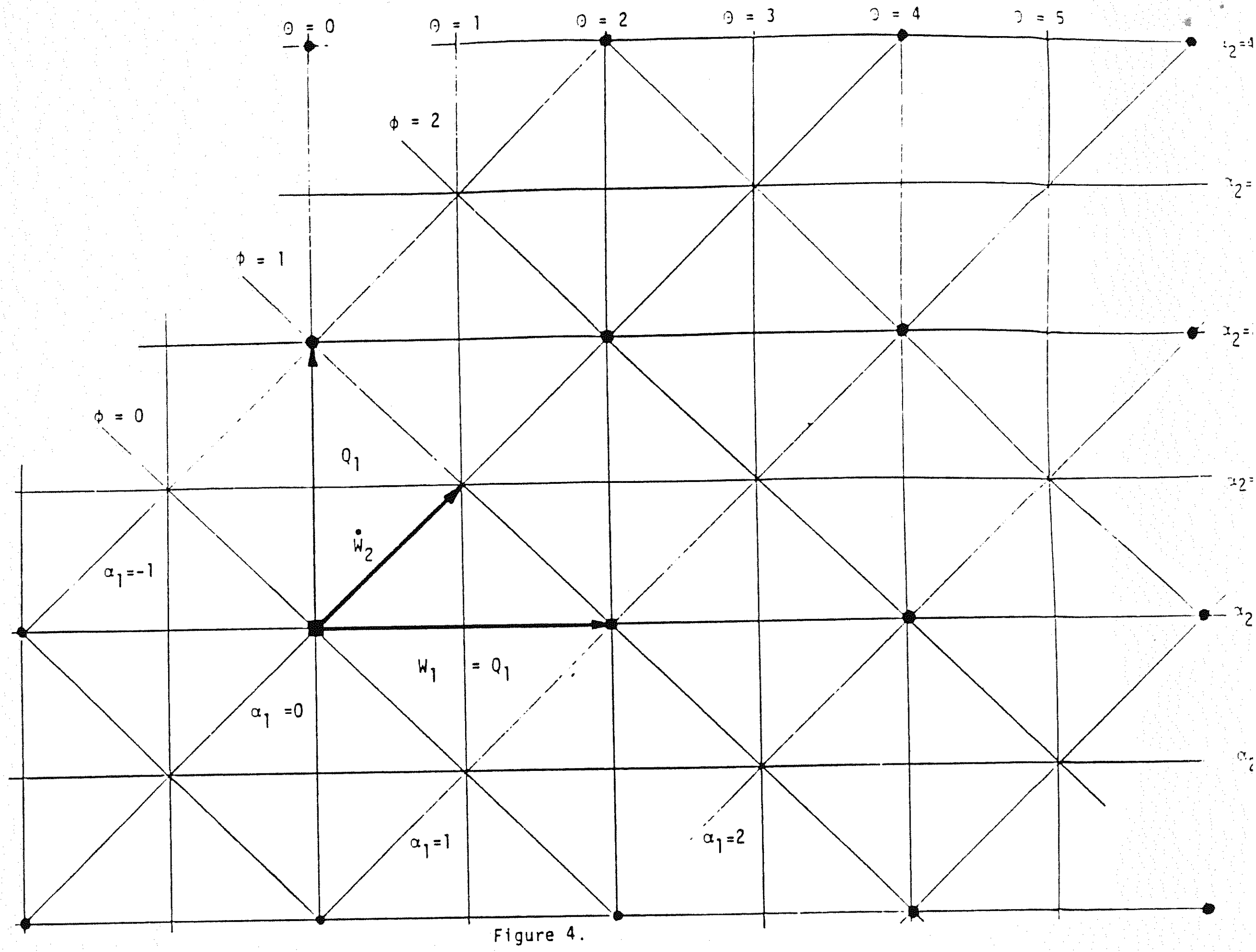

(1,1,-2)
Θ = 1
Θ = 2
Θ=3
Θ = 4
α_2 = 2
(2,0,-2)
Θ = 0
Q_2
α_2 = 1
(1,0,-1)
Θ = -1
$\mathring{W}_2$
α_2 = 0
0
$\mathring{W}_1$
(2,-1,-1)
Q_1
α_2 = -1
(-1,0,1)
α_1 = 0
α_1= 1
α_1= 2

Figure 3.



Θ = 0
Θ = 1
Θ = 2
Θ = 3
Θ = 4
Θ = 5
ϕ = 2
ϕ = 1
ϕ = 0
Q_1
$\dot{W}_2$
α_1=-1
$W_1 = Q_1$
α_1 =0
α_1=1
α_1=2

Figure 4.

Centre	Group	$\mathring{W}$ as matrix	$\mathring{W}$ as weight	$\mathring{W}$ -representation	Highest root expansion
$\mathbb{Z}_2$	Spin (2n+1)	$\sigma_{\mu\nu} = \frac{i}{4}\left[\gamma_\mu, \gamma_\nu\right]$ any $\mu \neq \nu$	ω_1	F_1 (vector)	$\alpha_1 + 2\alpha_2 + .. + 2\alpha_{r-1} + 2\alpha_r$
	Sympl(2n)	$(1/2)\sigma_3 \times \mathbb{1}_n$	$2\,\omega_r$	F_r (rank r atisymm. tensor)	$2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{r-1} + \alpha_r$
	E_7	$(1/2)\sigma_2 \times \mathbb{1}_{28}$	ω_1	56 -dimensional	$\alpha_1 +$
$\mathbb{Z}_3$	E_6	$(1/3)\ y \times \mathbb{1}_q$	$\omega_1, \ \omega_2$	$27, \ \overline{27}$	$\alpha_1 + \ \alpha_2 +$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	Spin (4n)	$\sigma_{\mu\nu}, (1/2)(1 \pm \gamma)\ \sigma_{\mu\nu}$	$\omega_1, \ \omega_{r-1}, \ \omega_r$	F_1 (vector), $S^{\pm}$ (spinor)	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$
$\mathbb{Z}_4$	Spin(4n+2)	$(1/2)\gamma, \ \frac{1}{4}\gamma + (1/2)(1 \pm \gamma)\sigma_{\mu\nu}$			
$\mathbb{Z}_n$	SU(n)	(1/n) diag(k, n-k)	ω_k, k = 1,...n-1	n-1 primitive reps. F_k	$\alpha_1 + \alpha_2 + \ldots + \ \alpha_{n-2} + \alpha_{n-1}$

Table 1.